(above) any neutral stability core are predicted regions of linear instability, areas outside (below) the curve are regions of linear stability. The primary mode of oscillation is the first transverse mode characterized by a dimensionless acoustic frequency (ω_0) of 1.8413. The ratio ω/ω_0 is shown as a parameter along the stability limit curves. When the lined configuration is evaluated, the liner is taken to cover one third of the available wall area and to be located at the injector end. A specific acoustic impendance K, ($K=1/\gamma\beta$) of 5.0 is used to characterize the linear damping.

As can be seen in the figure, for either the lined or unlined case, the stability curve for the continuous model is displaced upward from the corresponding curve for the discontinuous (multi-zone) model. This implies that the discontinuous model predictions overestimates the regions of linear stability, and underestimates the linearly stable regions on the n,τ plane.

Apart from this relative displacement effect the curves for the two model are qualitatively quite similar. In both cases the addition of a liner causes a substantial stabilizing effect (upward shift of the neutral stability curve) relative to the unlined configuration. The minimum point on the curves in all cases occurs at a frequency close to ω_0. Finally it should be noted that the stabilizing displacement of the curve with liner addition is of the same order of magnitude as the stabilizing shift upward caused by using the improved model. This implies that a damping effect of the same order of magnitude as that produced by the liner is ignored in the multi-zone work.

Figure 2 is a similar type of plot for a different steady-state combustion distribution. The distribution in this case is one predicted using a vaporization controlled mass generation model. This distribution is described more fully in Ref. 1. Only lined configurations are considered in Fig. 2, with $L/R=2.0$, $K=10.0$, $M_\xi=0.1984$, and $\omega_0=1.8413$. The upward displacement of the stability limit curve for the continuous model relative to the discontinuous model, as well as a general qualitative similarity of the curves, is present also for this distribution.

With the results previously given in mind the question naturally arises as to what is neglected in the multi-zone model which accounts for its substantial underestimation of combustor stability. The answer can be found in a term which is present on the right-hand side of the momentum equation which exerts a considerable damping effect. Specifically, this term is $Q'\ (\bar{u}_\ell-\bar{u})/\bar{\rho}$, which is associated with the acceleration a propellant element experiences as it is generated. This is a source dependent term and does not appear in the differential equations for the multi-zone model. Neither is it accounted for in the matching (boundary) conditions between successive source free zones. As a result this important damping effect is lost in the multi-zone results and renders the predictions of that approach substantially and systematically in error.

Reference

[1] Baer, M. R., Mitchell, C. E., and Espander, W. R., "Stability of Partially Lined Combustors with Distributed Combustion," *AIAA Journal*, Vol. 12, April 1974, pp. 475-480.

Frequency Spectrum of Shells

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It is customary to neglect certain lower order terms in the formulation of thin-shell equations. The classical equation of motion for thin shells are given in textbooks.[1] The terms neglected normally, do not give rise to significant errors in the values of the natural frequencies when the thickness is small. However, when the thickness is large, the terms can change values of the natural frequencies significantly and also can modify the frequency spectrum. This Note is intended to show how the frequency spectrum gets modified when these lower order terms are included.

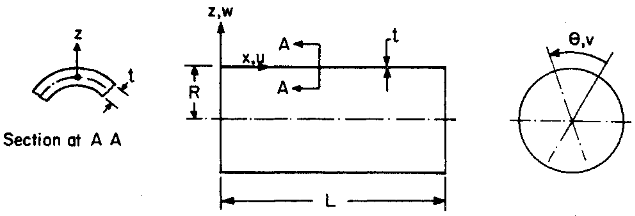

Fig. 1 Cylindrical shell.

Figure 1 shows a cylindrical shell of length L, radius R, and thickness t. Using the classical assumptions of normals to the middle surface before deformation remaining normal to the middle surface after deformation and assuming negligible shear strains, we have the strain displacements relationships in the form

$$\epsilon_{xx}=u,_x-z\ w,_{xx} \tag{1a}$$

$$\epsilon_{\theta\theta}=\frac{1}{R}\,v,_\theta+\frac{1}{R+z}\,w-\frac{z}{(R+z)R}\,w,_{\theta\theta} \tag{1b}$$

$$\epsilon_{x\theta}=\frac{R+z}{R}\,v,_x-\frac{2R+z}{R(R+z)}\,z\ w,_{x\theta}+\frac{u,_\theta}{R+z} \tag{1c}$$

The subscripts, θ, and, x indicate partial derivatives with respect to θ and x, respectively, and xx, etc., represent differentiation twice. The stress-strain relationships are

$$\sigma_{xx}=E'\ (\epsilon_{xx}+\nu\epsilon_{\theta\theta}) \tag{2a}$$

$$\sigma_{\theta\theta}=E'\ (\epsilon_{\theta\theta}+\nu\epsilon_{xx}) \tag{2b}$$

$$\sigma_{x\theta}=G\ \epsilon_{x\theta} \tag{2c}$$

with

$$E'=E/(1-\nu^2) \tag{3}$$

where E is the Young modulus, G is the shear modulus and ν is the Poisson ratio.

The displacement at any point in the shell $\bar{u}$, $\bar{v}$, $\bar{w}$ can be expressed in terms of the midplane displacements as

$$\bar{u}=u-z\ w,_x \qquad \bar{v}=v-z(w,_\theta-v)/R \qquad \bar{w}=w \tag{4}$$

Assuming sinusoidal oscillations, the maximum kinetic energy may be written as

$$T=\frac{\omega^2\rho}{2}\iiint(\bar{u}^2+\bar{v}^2+\bar{w}^2)\ (R+z)\mathrm{d}x\ \mathrm{d}z\ \mathrm{d}\theta \tag{5}$$

and the maximum strain energy U_e as

$$U_e=\tfrac{1}{2}\iiint(\sigma_{xx}\epsilon_{xx}+\sigma_{\theta\theta}+\sigma_{x\theta}\epsilon_{x\theta})\ (R+z)\mathrm{d}x\ \mathrm{d}z\ \mathrm{d}\theta \tag{6}$$

where ω is the natural frequency and ρ is the mass density of the shell. Using Eqs. (1-4) in Eqs. (5) and (6) and utilizing the principle of stationarity of the Lagrangian, the dynamical equations of motion may be obtained as

$$u,_{xx}+A_1u,_{\theta\theta}+A_2v,_{x\theta}+A_3w,_x+A_4u$$
$$+[A_5w,_{xxx}+A_6w,_{x\theta\theta}+A_7w,_x]=0 \tag{7a}$$

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Table 1 Differences in the coefficients

Constant	Including lower order terms	Classical theory[1]
A_1	$[(1-\nu)\log[(1+t/2R)/(1-t/2R)]]/2Rt$	$(1-\nu)/2R^2$
B_2	$(1-\nu)P/2Rt$	$(1-\nu)/2$
B_5	$\omega^2\rho P/E'Rt$	$\omega^2\rho/E'$
C_3	Q/R^2t	I/R^4t
C_6	T/Rt	$1/R^2$

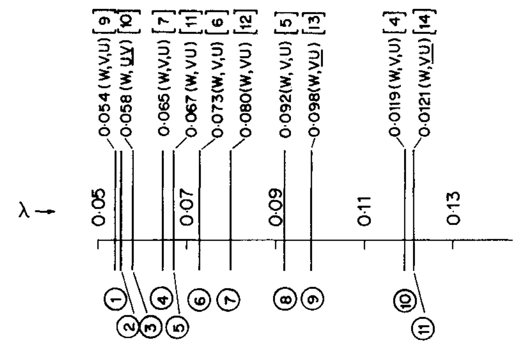

Fig. 2 Frequency spectrum for the case $L/R=0.5$, $R/t=1000$.

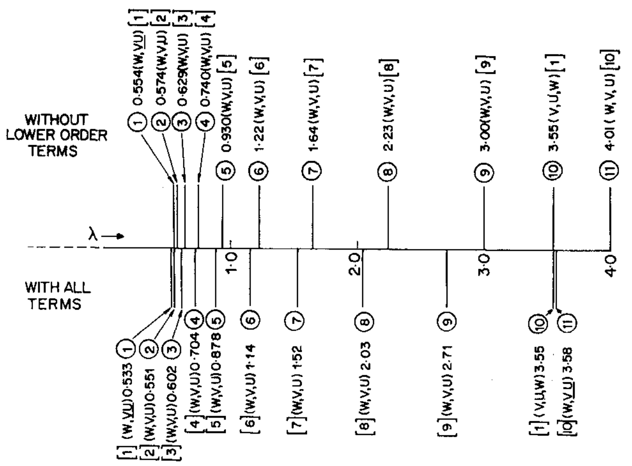

Fig. 3 Frequency spectrum for the case $L/R=0.5$, $R/t=100$.

$$B_1u,_{x\theta}+B_2v,_{xx}+B_3v,_{\theta\theta}+B_4w,_{\theta}$$

$$+B_5v+[B_6w,_{xx\theta}+B_7w,_{\theta}]=0 \qquad (7b)$$

$$C_1w,_{xxxx}+C_2w,_{xx\theta\theta}+C_3w,_{\theta\theta\theta\theta}+C_4u,_{x}$$

$$+C_5v,_{\theta}+(C_6+C_7)w+[C_8u,_{xxx}+C_9v,_{xx\theta}$$

$$+C_{10}u,_{x\theta\theta}+C_{11}u,_{x}+C_{12}v,_{\theta}+C_{13}w,_{xx}+C_{14}w,_{\theta\theta}]=0 \qquad (7c)$$

The lower order terms usually ignored are the bracketed terms in Eq. (7). The coefficients of most of the other terms are the same as in the classical equations, excepting a few. These differences are presented in Table 1.

For a simply supported shell, it is possible to obtain the exact solution taking the displacements in the form

$$u=U\cos(\pi x/L)\cos n\theta \qquad (8a)$$

$$v=V\sin(\pi x/L)\sin n\theta \qquad (8b)$$

$$w=W\sin(\pi x/L)\cos n\theta \qquad (8c)$$

and using the Galerkin technique. In Eqs. (8) U, V, W represent amplitudes, L is half of the longitudinal wavelength and n is the circumferential wave number. The frequency

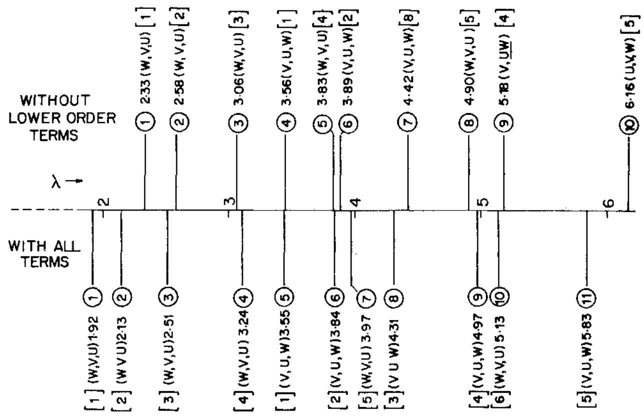

Fig. 4 Frequency spectrum for the case $L/R=0.5$, $R/t=4$.

spectrum (first 10 frequencies) obtained for three configurations of cylindrical shell are given in Figs. 2-4. In these figures, the values of the frequency parameter λ, defined as, $\lambda=\omega^2\rho L^2/E(1-\nu^2)$, the circumferential wave number (in square brackets), the frequency number (circled numbers), the mode shapes as (W, U, V), etc., are given. The mode shape when given as (W,U,V) means $W>U>V$ and when given as (W,UV) mean $W>U$ or V and U and V are of the same order.

Figure 2 shows the frequency spectrum of a very thin shell. All the first 10 modes involve primarily translational motion in z direction. The fundamental has a circumferential wave number of 9. The lower-order terms, do not significantly alter the frequency parameters and therefore are not reported here. Figure 5 shows the frequency spectrum of a moderately thin shell. From this figure, it is clear that with the inclusion of these terms, the values of frequency changes. For example, in the 8th frequency, there is a difference of about 7% and this difference increases (decreases) as the mode number increases (decreases). In Figure 4, the results are given for the case of a thick shell. This emphasizes the change of not only of the value of the frequency but also of the order of the mode. For example, the 4th mode, when lower order terms are neglected, is of the type $V>U>W$, whereas when these terms are included, this of the type $W>V>U$. We also observe such changes in 5th, 7th, and 8th, and 10th modes.

It should be mentioned here that the secondary effects such as transverse shear may modify the frequency spectrum further. These effects will be significant in the case of thick shells and so it will be necessary to include them also for correct prediction of the frequency spectrum of thick shells. In order to bring out the nature and extent of the influence of the lower order terms on the frequency spectrum, the effect of transverse shear is excluded from this analysis. The results reported in Figs. 2-4 show the influence of these terms only and bring out the need to include these terms also in addition to other secondary effects, for correct prediction of the frequency spectrum of thick shells. With modern constructions, such as layered and sandwich shells, it is possible that these lower order terms assume greater significance and influence the dynamic response analysis of such structures.

Appendix: Constants in the Governing Equations

$$A_1=\frac{(1-\nu)T}{2RT}, \quad B_1=\frac{1+\nu}{2R}, \quad C_1=\frac{I}{t},$$

$$A_2=\frac{1+\nu}{2R}, \quad B_2=\frac{(1-\nu)P}{2Rt}, \quad C_2=\frac{2I}{R^2t}$$

$$A_3=\nu/R, \quad B_3=B_4=1/R^2, \quad C_3=\frac{Q}{R^2t}$$

$$A_4=\frac{(1-\nu^2)\omega^2\rho}{E}, \quad B_5=\frac{\omega^2\rho P(1-\nu^2)}{ERt}, \quad C_4=\frac{\nu}{R}$$

$$A_5 = -I/Rt, \quad B_6 = -(3-\nu)I/2R^2t, \quad C_5 = 1/R^2$$

$$A_6 = \frac{(1-\nu)(RT-t)}{2\,Rt}, \quad B_7 = -\frac{2\omega^2\rho I(1-\nu)}{E\,R^2t}$$

$$A_7 = -\frac{\omega^2\rho I(1-\nu^2)}{E\,R\,t}, \quad C_7 = -\frac{(1-\nu^2)\omega^2\rho}{E}$$

$$C_8 = -I/Rt, \quad C_9 = -(3-\nu)I/2R^2t, \quad C_{10} = \frac{Q(1-\nu)}{2\,R\,t}$$

$$C_{11} = -I\omega^2\rho(1-\nu^2)/ERt,$$

$$C_{12} = -2I\omega^2\rho(1-\nu^2)/ER^2t$$

$$C_{13} = \omega^2\rho I(1-\nu^2)/Et, \quad C_{14} = \frac{2Q}{R^2t} + \frac{I\omega^2\rho(1-\nu^2)}{R^2t\,E}$$

where

$$I = t^3/12$$

$$T = \log(1+t/2R)//1-t/2R)$$

$$P = Rt[1+(t/2R)^2]$$

$$Q = (RT-t)$$

References

[1] Kraus, H., *Thin Elastic Shells,* Wiley, New York, 1967, p. 297.

Properties of Axial or Torsional Free-Vibration Frequency of Rods

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Introduction

IT is well known that the free-vibration frequencies of a system with a finite number of degrees of freedom are given by the roots of the equation obtained by requiring the determinant of the dynamic matrix to vanish.[1] In the case of longitudinal or torsional vibrations of rods approximated by finite elements, the dynamic matrix is tridiagonal and possesses a repetitive structure, which enables one to show that the eigenvalues have certain interesting properties. These properties were observed by Johnson[2] while investigating minimum-weight rods under dynamic loads.

It will be shown that for a clamped rod with an odd number of degrees of freedom, the middle frequency is independent of any nonuniformity in the area distribution. Then it will be shown that for any rod, the frequencies in the lower half of its spectrum are conjugate to the frequencies in the upper half in the sense that if the design is modified in a way such as to maintain unchanged some frequency in the lower half spectrum, its conjugate frequency in the upper half will also remain unchanged, and finally, upper bounds on the highest frequency of both the lower half and entire spectra will be given.

For brevity, a rod with an off number of degrees of freedom (that is, a rod represented by an odd number of finite elements, Fig. 1) will be called an "odd rod," and an "even rod," is one with an even number of degrees of freedom. Although the analysis is carried out for longitudinal vibrations, it can be applied to torsional vibrations the same way.

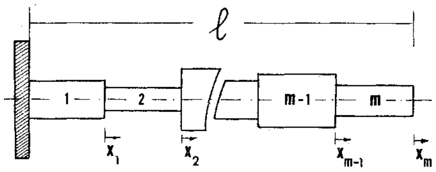

Fig. 1 Discretization of the rod in *m* finite elements of equal length.

Equations of Motion of Vibrating Rod

The element stiffness and inertia matrices can be written, respectively, as[1]

$$[K_i] = \frac{E}{\ell_i}\begin{bmatrix} A_i & -A_i \\ -A_i & A_i \end{bmatrix}$$

$$[I_i] = \frac{\rho\ell_i}{6}\begin{bmatrix} 2A_i & A_i \\ A_i & 2A_i \end{bmatrix} \tag{1}$$

where: A_i = cross sectional area of i-th element; ℓ_i = length of i-th element; E = Young's modulus; and ρ = specific mass. If the rod is divided into m elements of equal length ℓ/m, with the degrees of freedom numbered sequentially as in Fig. 1, and the left end is assumed fixed, the equations of motion for harmonic oscillations with frequency ω reduce to

$$\begin{bmatrix} (A_1+A_2)\delta_1 & -A_2\delta_2 & & & & \\ -A_2\delta_2 & (A_2+A_3)\delta_1 & -A_3\delta_2 & & & \\ & & (A_3+A_4)\delta_1 & -A_4\delta_2 & & \\ & & & -A_{m-1}\delta_2 & (A_{m-1}+A_m)\delta_1 & -A_m\delta_2 \\ & & & & -A_m\delta_2 & A_m\delta_1 \end{bmatrix}\{x\} = 0 \tag{2}$$

where: $\{x\}$ is a vector of axial displacements, and

$$\delta_1 = 1-(\omega^2\rho\,\ell^2/3Em^2) \qquad \delta_2 = 1+(\omega^2\rho\ell^2/6Em^2)$$

Property of Middle Frequency Invariance and Conjugate Pairs

The condition for the existence of nontrivial solution of Eq.

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